Conjugacy and Sylow Theorems - Gallian, Ch 24 # 4,5, Fraleigh, Ch 4.2 # 1-6

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1 Gallien

(4) Describe the conjugacy classes of an abelian group.

For G, an abelian group, and $x, a \in G$

$$cl(a) = \{xax^{-1}\}$$
$$cl(a) = \{(xx^{-1})a\}$$
$$cl(a) = \{a\}$$

so the conjugacy classes of an abelian group are the elements of that group.

(5) Exhibit a Sylow 2-subgroup of S_4. Describe an isomorphism from this group to D_4.

$$|S_4| = 4! = 24$$

$2^3 = 8$ divides 24, but $2^4 = 16$ doesn't, so the order of the Sylow 2-subgroup is 8. By the Sylow's third theorem, there are either 1 or 3 Sylow 2-subgroups.

$D_4 = \{R_0, R_{90}, R_{180}, R_{270}, H, V, D_1, D_2\}$

$H \cong D_4$		R_0	R_{90}	R_{180}	R_{270}	H	V	D_1	D_2
		e	(1432)	(13)(24)	(1234)	(13)	(24)	(14)(32)	(12)(34)
R_0	e	e	(1432)	(13)(24)	(1234)	(13)	(24)	(14)(32)	(12)(34)
R_{90}	(1432)	(1432)	(13)(24)	(1234)	e	(12)(34)	(14)(32)	(13)	(24)
R_{180}	(13)(24)	(13)(24)	(1234)	e	(1432)	(24)	(13)	(12)(34)	(14)(23)
R_{270}	(1234)	(1234)	e	(1432)	(13)(24)	(14)(32)	(12)(34)	(24)	(13)
H	(13)	(13)	(14)(23)	(24)	(12)(34)	e	(13)(24)	(1432)	(1234)
V	(24)	(24)	(12)(34)	(13)	(14)(23)	(13)(24)	e	(1234)	(1432)
D_1	(14)(32)	(14)(32)	(24)	(12)(34)	(13)	(1234)	(1432)	e	(13)(24)
D_2	(12)(34)	(12)(34)	(13)	(14)(23)	(24)	(1432)	(1234)	(13)(24)	e

where e is the identity element in S_4, (1)(2)(3)(4).

2 Fraleigh

1. A Sylow 3-subgroup of a group of order 12 has order 3. $3^1 = 3$ divides 12, but $3^2 = 9$ does not.

2. A Sylow 3-subgroup of a group of order 54 has order 27. $3^3 = 27$ divides 54, but $3^4 = 81$ does not.

3. A group of order 24 must have either 1 or 3 Sylow 2-subgroups. The Sylow 2-subgroups have order $2^3 = 8$ because $2^4 = 16$ doesn't divide 24. The number of Sylow 2-subgroups must be equal to 1 modulo 2 and must divide 24. $1 \equiv 3 \equiv 5 \equiv \ldots \equiv 23$ and of this list, only 1 and 3 divide 24.

4. By the same reasoning, a group of order 255 ($= 3 \times 5 \times 17$) must have either 1 or 85 Sylow 3-subgroups and 1 or 51 Sylow 5-subgroups.

5. Find all Sylow 3-subgroups of S_4 and demonstrate that they are all conjugate.

   By Sylow's third theorem, S_4 has either 1 or 4 Sylow 3-subgroups. Each Sylow 3-subgroup must have order 3.

$$A = \{(123), (132), e\}$$
$$B = \{(134), (143), e\}$$
$$C = \{(234), (243), e\}$$
$$D = \{(124), (142), e\}$$

   These are all 4 Sylow 3-subgroups of S_4. They are all cyclic groups of order 3.

   Now we check conjugacy. We show that B, C, D are all conjugate to A, so all four groups are conjugate to all the others.

$$A \cong B : (234)A(243) = \{(234)(123)(243), (234)(132)(243), (234)e(243)\} = \{(134), (143), e\} = B$$
$$A \cong C : (134)A(143) = \{(134)(123)(143), (134)(132)(143), (134)e(143)\} = \{(134), (143), e\} = C$$
$$A \cong D : (243)A(234) = \{(243)(123)(234), (243)(132)(234), (243)e(234)\} = \{(142), (124), e\} = D$$

6. Find two Sylow 2-subgroups of S_4 and show that they are conjugate.

$$J = \{e, (1432), (13)(24), (1234), (13), (24), (14)(32), (12)(34)\}$$
$$K = \{e, (1243), (14)(23), (2134), (14), (23), (12)(43), (13)(24)\}$$

   J and K are conjugate if there exists some $x \in S_4$ such that $xJx^{-1} = K$. (12) is one such x.

$$(12)J(12)^{-1}$$

$$= \{(12)e(12), (12)(1432)(12), (12)(13)(24)(12), (12)(1234)(12), (12)(13)(12), (12)(24)(12), (12)(14)(32)(12), (12)(12)(34)(12)\}$$

$$= \{e, (1234), (14)(32), (1342), (23), (14), (13)(24), (12)(34)\} = K$$

3 Extra

Theorem: Conjugacy is an equivalence relation.

Proof:

1. Reflexive: We must show that $\exists x$ such that $xax^{-1} = a$. We can choose a here, because

   $$aaa^{-1} = ae = a$$

   where e is the identity. We could also have chosen e, because

   $$eae^{-1} = eae = a$$

2. Symmetric:

   $$\begin{aligned} xax^{-1} &= b \\ xa &= bx \\ a &= x^{-1}bx \end{aligned}$$

   so if we let $x^{-1} = y \in G$ (G is a group), the above becomes

   $$a = yby^{-1}$$

   so $a \cong b \Rightarrow b \cong a$.

3. Transitive: Assume that $xax^{-1} = b$ and $yby^{-1} = c, x, y \in G$. Then, $b = y^{-1}cy$. As above, we let $y^{-1} = z \in G$ so that $b = zcz^{-1}$. Then

   $$\begin{aligned} xax^{-1} &= zcz^{-1} \\ a &= (x^{-1}z)c(z^{-1}x) \end{aligned}$$

   We substitute again, letting $x^{-1}z = w \in G$, so that $w^{-1} = z^{-1}x$, giving

   $$a = wcw^{-1}$$

   so $c \cong a$.