

## Notebook Problem 5.6.34

Jason Wojciechowski

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**Theorem:** Let  $f(x) \in F[x]$  where  $F$  is a field, and let  $\alpha \in F$ . Then the remainder  $r(x)$  when  $f(x)$  is divided by  $x - \alpha$  is  $f(\alpha)$ .

**Proof:** When we use long division to divide  $x - \alpha$  into the polynomial  $a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ , the first term in the quotient is  $a_n x^{n-1}$ . When this term is multiplied by  $\alpha$  and subtracted from  $a_{n-1} x^{n-1}$ , we're left with  $(a_{n-1} + \alpha) x^{n-1}$ , so that the next term in the quotient is  $(a_{n-1} + \alpha) x^{n-2}$ . We multiply this by  $\alpha$  and subtract the result from the next term in the numerator, leaving us with  $(a_{n-2} + \alpha a_{n-1} + \alpha^2) x^{n-2}$ .

Thus, every time we multiply a term in the quotient by  $\alpha$  and subtract from the next term in the numerator, we multiply the previous remainder by  $\alpha$  and then add a constant  $a_k$  term. This whole expression is, of course, multiplied by some  $x^k$ , but the exponent in that term is decreasing, so it eventually reaches  $x^0 = 1$ . At that point, we will just have multiplied  $a_1 + a_2 \alpha + a_3 \alpha^2 + \dots + a_n \alpha^{n-1}$  by  $\alpha$  and added an  $a_0$ , leaving us with  $a_0 + a_1 \alpha + a_2 \alpha^2 + \dots + a_{n-1} \alpha^{n-1} + a_n \alpha^n$ , which is  $f(\alpha)$ .