Notebook Problem 4.8 (Stewart)

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Theorem: Let $\mathbf{A}$ be the field of algebraic numbers and $\mathbf{Q}$ be the field of rational numbers. Then $[\mathbf{A} : \mathbf{Q}] = \infty$.

Proof: We'll use Eisenstein's criterion to show that there exist irreducible polynomials over $\mathbf{Q}$ of arbitrarily large degree, thus showing that $[\mathbf{A} : \mathbf{Q}] = \infty$.

Take $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ for any $n \in \mathbf{N}$. Now choose a $q > 1$ such that $q \not| a_n$ (we know this can always be found since we can just choose a q greater than a_n) and set $a_0, \ldots, a_{n-1}$ equal to q. Then $q + qx + qx^2 + \ldots + a_n x^n$ is irreducible by Eistenstein's criterion since $q \not| a_n$, $q \mid a_i, 0 \leq i \leq n-1$, and $q^2 \not| a_0$, since $q^2 \not| q$.