

Frleigh 9.1.37

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Lemma 1. *An automorphism of a field E carries elements that are squares of elements in E onto elements that are squares of elements in E .*

Proof. Suppose $\phi: E \rightarrow E$ is an automorphism. Then for every $a_k \in E$ such that $\phi(a_k) = a_m$, $\phi(a_k a_k) = a_m a_m$, so all $a_k a_k$'s, or all squares in E , go to another square in E . \square

Lemma 2. *An automorphism of the field \mathbb{R} carries positive numbers onto positive numbers.*

Proof. Since all positive $r \in \mathbb{R}$ have two real square roots, we can say that a real number is a square if and only if it is positive. Thus, since squares get carried to squares in any field (Lemma 1), positive numbers get carried to positive numbers in the real numbers. \square

Lemma 3. *If ϕ is an automorphism of \mathbb{R} and $a < b$, $a, b \in \mathbb{R}$, then $\phi(a) < \phi(b)$.*

Proof. Suppose $b > a$. Then $b - a > 0$, so, by Lemma 2, $\phi(b - a) > 0$, so $\phi(b) - \phi(a) > 0$, so $\phi(b) > \phi(a)$. \square

Theorem 1. *The only automorphism of \mathbb{R} is the identity automorphism.*

Proof. The image of 1 under any automorphism must be 1, since it is the multiplicative identity. Then, since automorphisms respect the addition operation, every integers must be taken to itself by an automorphism on the reals. Since automorphisms must also respect multiplication (and thus division), every rational must be taken to itself.

Now the only reals left in question are the irrationals. Every irrational number can be squeezed infinitely closely by two rationals. For example, π is between 3 and 4, 3.1 and 3.2, 3.14 and 3.15, and so on. By Lemma 3, an irrational which sits between rationals a and b must be taken to an irrational between a and b . Since we can squeeze every irrational infinitely closely with a sequence of rationals on either side of it, every irrational must end up being taken to itself. Thus, every real number must be taken to itself by an automorphism on the reals, so the only automorphism on the reals is the identity automorphism. \square