Notebook problem about an abelian Galois group

Jason Wojciechowski

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Theorem. *The Galois group of the splitting field of $x^n - a$ over the splitting field of $x^n - 1$ is abelian.*

Proof. Let $\omega \in K$ (where K is the splitting field of $x^n - 1$) be any zero of $x^n - 1$ and $\alpha \in L$ (where L is the splitting field of $x^n - a$) be any zero of $x^n - a$. Zeros of $x^n - a$ can also be expressed as $\omega\alpha$. Since the splitting field L of $x^n - a$ is $K(\alpha)$, the members of the Galois group of $L : K$, the automorphisms that fix K, are fully determined by their action on α.

Consider two automorphisms, ψ_1 and ψ_2 such that $\psi_1(\alpha) = \omega_1\alpha$ and $\psi_2(\alpha) = \omega_2\alpha$, where the ω_i's are zeros of $x^n - 1$. Let us examine $\psi_1 \circ \psi_2(\alpha)$.

$$\psi_1 \circ \psi_2(\alpha) = \psi_1(\omega_2\alpha) = \omega_2\omega_1\alpha = \omega_1\omega_2\alpha = \psi_2(\omega_1\alpha) = \psi_2 \circ \psi_1(\alpha).$$

ω_1 commutes with ω_2 because $e^{\frac{a\cdot 2\pi i\theta}{n}} e^{\frac{b\cdot 2\pi i\theta}{n}} = e^{\frac{(a+b)2\pi i\theta}{n}} = e^{\frac{b\cdot 2\pi i\theta}{n}} e^{\frac{a\cdot 2\pi i\theta}{n}}$. Thus we have that the Galois group is abelian. □